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The Lebesgue Constant for Higher Order Hermite–Fejér Interpolation on the Chebyshev Nodes

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For a fixed integer $m \geqslant 0$, and for $n = 1, 2, 3, \ldots$, let $\lambda_{2m,n}(x)$ denote the Lebesgue function associated with $(0, 1, \ldots, 2m)$ Hermite-Fejér polynomial interpolation at the Chebyshev nodes $\{\cos[(2k-1)\pi/(2n)]: k = 1, 2, \ldots, n\}$. We examine the Lebesgue constant $\Lambda_{2m,n} := \max\{\lambda_{2m,n}(x): -1 \leqslant x \leqslant 1\}$, and show that $\Lambda_{2m,n} = \lambda_{m,n}(1)$, thereby generalising a result of H. Ehlich and K. Zeller for Lagrange interpolation on the Chebyshev nodes. As well, the infinite term in the asymptotic expansion of $\Lambda_{2m,n}$ as $n \to \infty$ is obtained, and this result is extended to give a complete asymptotic expansion for $\Lambda_{2,n}$.

1. INTRODUCTION

Suppose f is a continuous real-valued function defined on the interval $[-1, 1]$, and let

$$X = \{x_{k,n}: k = 1, 2, \ldots, n, \ n = 1, 2, 3, \ldots\} \tag{1.1}$$

be a triangular matrix such that, for all n,

$$1 \geqslant x_{1,n} > x_{2,n} > \cdots > x_{n,n} \geqslant -1. \tag{1.2}$$

Then, for each integer $m \geqslant 0$, there exists a unique polynomial $H_{m,n}(X, f, x)$ of degree at most $(m+1)n - 1$ which satisfies

$$H^{(t)}_{m,n}(X, f, x_{k,n}) = \delta_{0,t} f(x_{k,n}), \qquad 1 \leqslant k \leqslant n, \quad 0 \leqslant t \leqslant m.$$

$H_{m,n}(X, f, x)$ is referred to as the $(0, 1, \ldots, m)$ Hermite–Fejér (HF) interpolation polynomial of $f(x)$, and it can be written as

$$H_{m,n}(X, f, x) = \sum_{k=1}^{n} f(x_{k,n}) A_{k,m,n}(X, x),$$

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where $A_{k,m,n}(X, x)$ is the unique polynomial of degree at most $(m+1)n-1$ which satisfies

$$A^{(t)}_{k,m,n}(X, x_{j,n}) = \delta_{0,t}\delta_{k,j}, \qquad 1 \leqslant k, j \leqslant n, \quad 0 \leqslant t \leqslant m. \tag{1.3}$$

Note that $H_{0,n}(X, f, x)$ is the well-known Lagrange interpolation polynomial of $f(x)$.

We consider the uniform norm $\|f\| := \max_{-1 \leqslant x \leqslant 1} |f(x)|$ on $C[-1, 1]$. The norm of the linear operator $H_{m,n}(X, \cdot, \cdot)\colon C[-1, 1] \to C[-1, 1]$ defined by $H_{m,n}(X, \cdot, \cdot)(f)(x) = H_{m,n}(X, f, x)$, with respect to the uniform norm, will be denoted by $\Lambda_{m,n}(X)$. This quantity is known as the Lebesgue constant of order n for $(0, 1, ..., m)$ HF interpolation on X, and is given by

$$\Lambda_{m,n}(X) = \max_{-1 \leqslant x \leqslant 1} \lambda_{m,n}(X, x),$$

where

$$\lambda_{m,n}(X, x) := \sum_{k=1}^{n} |A_{k,m,n}(X, x)| \tag{1.4}$$

is the Lebesgue function of order n for $(0, 1, ..., m)$ HF interpolation on X.

For Lagrange interpolation, it is known (c.f. Rivlin [12, Section 1.3]) that there exists a positive constant c such that

$$\Lambda_{0,n}(X) > \frac{2}{\pi} \log n + c, \qquad n = 1, 2, 3, ..., \tag{1.5}$$

for any X. A consequence of (1.5) is the classic result, due to Faber [6], that for any matrix X, there exists $f \in C[-1, 1]$ so that $H_{0,n}(X, f, x)$ does not tend uniformly to $f(x)$ on $[-1, 1]$ as $n \to \infty$. On the other hand, if T denotes the matrix of Chebyshev nodes

$$T = \left\{ \cos\left(\frac{2k-1}{2n}\pi\right)\colon k = 1, 2, ..., n;\ n = 1, 2, 3, ... \right\},$$

then

$$\Lambda_{0,n}(T) \leqslant \frac{2}{\pi} \log n + 1, \qquad n = 1, 2, 3, ... \tag{1.6}$$

(See Rivlin [12, Theorem 1.2].) The modulus of continuity $\omega(\delta; f)$ of f is defined by

$$\omega(\delta; f) = \max\{ |f(s) - f(t)| : \{s, t\} \subset [-1, 1], |s - t| \leqslant \delta \}.$$

It follows from (1.6) (c.f. Rivlin [11, Section 4.1]) that if $f \in C[-1, 1]$ satisfies the relatively weak condition $\omega(1/n; f) \log n \to 0$ as $n \to \infty$, then the sequence of Lagrange interpolation polynomials $H_{0,n}(T, f, x)$ converges uniformly to $f(x)$ on $[-1, 1]$ as $n \to \infty$. In view of these results, it can be seen that the Chebyshev nodes T are a good choice if uniform approximation by Lagrange interpolation polynomials is required.

For (0, 1, 2) HF interpolation, Szabados and Varma [20] showed that there is a constant $c_1 > 0$ so that for any system of nodes X,

$$\Lambda_{2,n}(X) \geqslant c_1 \log n.$$

This result was extended by Szabados [19], who proved that there are positive constants c_m so that

$$\Lambda_{2m,n}(X) \geqslant c_m \log n, \qquad m = 0, 1, 2, \ldots \tag{1.7}$$

Thus, for any system of nodes X, $H_{2m,n}(f, x)$ cannot converge uniformly to f for all $f \in C[-1, 1]$.

The problem of $(0, 1, \ldots, m)$ HF interpolation on the Chebyshev nodes (and on their generalization, the Jacobi nodes) has been studied by Sakai [13, 14], Vértesi [21, 22] and Sakai and Vértesi [15, 16]. In these papers it is shown that for each odd value of m, the Lebesgue constant $\Lambda_{m,n}(T)$ is bounded as $n \to \infty$, while if m is even,

$$\Lambda_{m,n}(T) = O(\log n), \qquad \text{as} \quad n \to \infty. \tag{1.8}$$

(Thus the order of magnitude on the right-hand side of (1.7) cannot be increased.) The aim of this paper is to extend the results of Sakai and Vértesi concerning $\Lambda_{2m,n}(T)$.

For Lagrange interpolation on the Chebyshev nodes, Ehlich and Zeller [5] have proved that

$$\Lambda_{0,n}(T) = \lambda_{0,n}(T, 1). \tag{1.9}$$

(See also Rivlin [12, Section 1.3] for a proof of (1.9), and Brutman [1] and Günttner [7] for closely related results.) This result was a key step in the process of finding a complete asymptotic expansion of $\Lambda_{0,n}(T)$. In this paper we generalise Ehlich and Zeller's result by proving the following theorem.

THEOREM 1. *For $m = 0, 1, 2, \ldots$, we have*

$$\Lambda_{2m,n}(T) = \lambda_{2m,n}(T, 1). \tag{1.10}$$

It follows from (1.4) and (1.10) that

$$\Lambda_{2m,n}(T)=\sum_{k=1}^{n}|A_{k,2m,n}(T,1)|.$$

By developing careful estimates for the $|A_{k,2m,n}(T,1)|$, we are able to improve (1.8) by establishing the following result.

THEOREM 2. *As* $n\to\infty$,

$$\Lambda_{2m,n}(T)=\frac{2}{\pi}\frac{(2m)!}{2^{2m}(m!)^2}\log n+O(1). \tag{1.11}$$

Thus the leading term in the asymptotic expansion of $\Lambda_{2m,n}(T)$ decreases with increasing m, and, indeed, behaves like $2\pi^{-3/2}m^{-1/2}\log n$ for large m.

In general it seems to be awkward to derive a complete asymptotic expansion for $\Lambda_{2m,n}(T)$. However, Shivakumar and Wong [18] have shown that $\Lambda_{0,n}(T)$ has the asymptotic expansion

$$\Lambda_{0,n}(T)=\frac{2}{\pi}\log n+\frac{2}{\pi}\left(\gamma+\log\frac{8}{\pi}\right)+\frac{8}{\pi}\sum_{s=1}^{r}\frac{(-1)^{s+1}A_s}{(2n)^{2s}}+\Phi_r(n), \tag{1.12}$$

where γ denotes the Euler-Mascheroni constant,

$$A_s=(2^{2s-1}-1)^2\frac{\pi^{2s}}{2s}\frac{B_{2s}^2}{(2s)!},\qquad s=1,2,3,...,$$

the B_s's are the Bernoulli numbers, and

$$0\leqslant(-1)^r\Phi_r(n)\leqslant\frac{8A_{r+1}}{\pi(2n)^{2r+2}},\qquad r=1,2,3,... \tag{1.13}$$

(Analogous results to those of Shivakumar and Wong have been developed independently by Dzjadyk and Ivanov [4] and Günttner [8].) In this paper we are able to derive the following asymptotic expansion for $\Lambda_{2,n}(T)$.

THEOREM 3. *The Lebesgue constant* $\Lambda_{2,n}(T)$ *can be written as*

$$\Lambda_{2,n}(T)=\frac{1}{\pi}\log n+\left(\frac{1}{\pi}\left(\gamma+\log\frac{8}{\pi}\right)+\frac{14}{\pi^3}\zeta(3)\right)+\left(\frac{\pi}{144}-\frac{1}{6\pi}\right)\frac{1}{n^2}+\frac{8}{\pi}\sum_{s=2}^{r}\frac{(-1)^s C_s}{(2n)^{2s}}+\Psi_r(n), \tag{1.14}$$

where $\zeta(k)$ *denotes the Riemann zeta function* $\zeta(k)=\sum_{n=1}^{\infty} n^{-k}$,

$$C_s=(2^{2s-1}-1)\frac{\pi^{2s}}{s}\frac{|B_{2s}|}{(2s)!}$$
$$\times\left(\frac{2s(2s-1)}{\pi^2}(2^{2s-3}-1)\,|B_{2s-2}|-\frac{(2^{2s-1}-1)}{4}|B_{2s}|\right), \tag{1.15}$$

(*so* $C_s>0$), *and*

$$|\Psi_r(n)|\leqslant\frac{8(2^{2r+1}-1)(2^{2r-1}-1)\,|B_{2r+2}|\,|B_{2r}|}{(r+1)(2r)!}\frac{\pi^{2r-1}}{(2n)^{2r+2}}, \qquad r=2, 3, \ldots \tag{1.16}$$

We will prove the above theorems in the following three sections of this paper. Also, for notational convenience, we will henceforth remove specific reference to T, and write $A_{k,m,n}(x)=A_{k,m,n}(T,x)$, $\lambda_{m,n}(x)=\lambda_{m,n}(T,x)$, $\Lambda_{m,n}=\Lambda_{m,n}(T)$, and

$$x_k=x_{k,n}=\cos\left(\frac{2k-1}{2n}\pi\right), \qquad k=1, 2, \ldots, n. \tag{1.17}$$

2. Proof of Theorem 1

Our proof of Theorem 1 is based on Ehlich and Zeller's method for establishing (1.9), as described in Rivlin [12, Section 1.3].

Let $\mathscr{T}_{M,1}$ denote the set of all trigonometric polynomials $T(\theta)$ which can be written in the form

$$T(\theta)=a_0+\sum_{k=1}^{M-1}(a_k\cos k\theta+b_k\sin k\theta)+a_M\cos M\theta, \tag{2.1}$$

and let $\mathscr{T}_{M,2}$ denote the set of all trigonometric polynomials $T(\theta)$ which can be written as

$$T(\theta)=a_0+\sum_{k=1}^{M-1}(a_k\cos k\theta+b_k\sin k\theta)+b_M\sin M\theta.$$

By a result of Cavaretta, Sharma and Varga [3], the problem of $(0, 1, 2, \ldots, 2m)$ Hermite interpolation by trigonometric polynomials on the nodes $(k\pi)/n$ $(k=0, 1, 2, \ldots, 2n-1)$ has a unique solution in the class

$\mathscr{T}_{n(2m+1),1}$. Thus, given numbers $c_{k,l}$ ($k=0, 1, ..., 2n-1$; $l=0, 1, ..., 2m$), there exists a unique $T(\theta) \in \mathscr{T}_{n(2m+1),1}$ such that

$$T^{(l)}\left(\frac{k\pi}{n}\right) = c_{k,l} \qquad 0 \leqslant k \leqslant 2n-1, \quad 0 \leqslant l \leqslant 2m.$$

(A different proof of the result of Cavaretta, Sharma and Varga is given by Sharma, Smith and Tzimbalario [17], and the results of both papers are described in Riemenschneider, Sharma and Smith [10, Theorem A].) Upon replacing θ with $\theta - \pi/(2n)$ in (2.1), it follows that for the nodes $(2k-1)\pi/(2n)$, ($k = 1, 2, ..., 2n$), the problem of $(0, 1, 2, ..., 2m)$ Hermite interpolation by trigonometric polynomials has a unique solution in the class $\mathscr{T}_{n(2m+1),2}$.

Now define

$$\theta_j = \theta_{j,n} = (2j-1)\frac{\pi}{2n}, \qquad j = 0, \pm 1, \pm 2, ...,$$

and let $m \geqslant 0$ be given. For $k = 1, 2, ..., 2n$, let $d_{k,2m}(\theta)$ be the unique element of $\mathscr{T}_{n(2m+1),2}$ which satisfies the $2n(2m+1)$ conditions

$$d^{(t)}_{k,2m}(\theta_j) = \delta_{0,t}\delta_{k,j}, \qquad 1 \leqslant j \leqslant 2n, \quad 0 \leqslant t \leqslant 2m. \tag{2.2}$$

Define

$$D_{2m,n}(\theta) = \sum_{k=1}^{2n} |d_{k,2m}(\theta)|, \tag{2.3}$$

and

$$\Delta_{2m,n} = \max\{D_{2m,n}(\theta) \colon 0 \leqslant \theta \leqslant 2\pi\}.$$

In the following three lemmas we develop a useful characterization of $\Delta_{2m,n}$.

LEMMA 1. $\Delta_{2m,n} = \max\{D_{2m,n}(\theta) \colon |\theta| \leqslant \pi/(2n)\}$.

Proof. Let

$$e_{k,2m}(\theta) = d_{k,2m}(\theta + \pi/n), \qquad k = 1, 2, ..., 2n.$$

The $e_{k,2m}$ are in $\mathscr{T}_{n(2m+1),2}$, and satisfy

$$e^{(t)}_{k,2m}(\theta_j) = d^{(t)}_{k,2m}(\theta_{j+1}) = \delta_{0,t}\delta_{k,j+1}, \qquad 0 \leqslant j \leqslant 2n-1, \quad 0 \leqslant t \leqslant 2m.$$

By the uniqueness properties of the $d_{k,2m}$, it follows that $e_{1,2m}(\theta) \equiv d_{2n,2m}(\theta)$ and $e_{k,2m}(\theta) \equiv d_{k-1,2m}(\theta)$ $(k = 2, 3, ..., 2n)$. That is,

$$\begin{cases} d_{1,2m}(\theta + \pi/n) \equiv d_{2n,2m}(\theta), \\ d_{k,2m}(\theta + \pi/n) \equiv d_{k-1,2m}(\theta), \qquad k = 2, 3, ..., 2n. \end{cases} \tag{2.4}$$

Hence, by (2.3),

$$D_{2m,n}(\theta) \equiv D_{2m,n}(\theta + \pi/n),$$

and so

$$\Delta_{2m,n} = \max\{D_{2m,n}(\theta)\colon |\theta| \leqslant \pi/(2n)\}.$$

This completes the proof of Lemma 1. ∎

Now define

$$t_{2m,n}(\theta) = \sum_{k=1}^{n} (-1)^{k-1} d_{k,2m}(\theta) - \sum_{k=n+1}^{2n} (-1)^{k-1} d_{k,2m}(\theta). \tag{2.5}$$

LEMMA 2. (i) $\Delta_{2m,n} = \max\{t_{2m,n}(\theta)\colon |\theta| \leqslant \pi/(2n)\}$;

(ii) *$t_{2m,n}(\theta)$ is an even function of θ.*

Proof. (i) Put

$$f_{k,2m}(\theta) = d_{k,2m}(2\theta_k - \theta), \qquad k = 1, 2, ..., 2n.$$

Each $f_{k,2m}$ is in $\mathscr{T}_{n(2m+1),2}$, and the $f_{k,2m}$ satisfy

$$f_{k,2m}^{(t)}(\theta_j) = (-1)^t d_{k,2m}^{(t)}(\theta_{2k-j}) = \delta_{0,t}\delta_{k,j}, \qquad 1 \leqslant j \leqslant 2n, \quad 0 \leqslant t \leqslant 2m.$$

Thus $f_{k,2m}(\theta) \equiv d_{k,2m}(\theta)$. That is,

$$d_{k,2m}(2\theta_k - \theta) \equiv d_{k,2m}(\theta). \tag{2.6}$$

Differntiating (2.6) $2m+1$ times, then putting $\theta = \theta_k$, gives

$$d_{k,2m}^{(2m+1)}(\theta_k) = 0,$$

while on putting $\theta = \theta_{k+n}$, and using $\theta_{k+n} = \theta_{k-n} + 2\pi$, we obtain

$$d_{k,2m}^{(2m+1)}(\theta_{k+n}) = 0.$$

Now, for each $k = 1, 2, ..., 2n$, $d'_{k,2m}(\theta)$ is a trigonometric polynomial of order $n(2m+1)$ or less which satisfies $d_{k,2m}^{(t)}(\theta_{k+r}) = 0$, $-(n-1) \leqslant r \leqslant n$, $1 \leqslant t \leqslant 2m$, and $d_{k,2m}^{(2m+1)}(\theta_k) = d_{k,2m}^{(2m+1)}(\theta_{k+n}) = 0$. Thus we have located

$2n-2$ zeros of $d'_{k,2m}(\theta)$ of multiplicity $2m$, and 2 zeros of multiplicity $2m+1$, in the interval $(\theta_{k-n}, \theta_{k+n}]$. Further, from $d_{k,2m}(\theta_j)=\delta_{k,j}$, $1 \leqslant j \leqslant 2n$, it follows from Rolle's Theorem that $d'_{k,2m}(\theta)$ has a zero in each of the intervals $(\theta_{k-r-1}, \theta_{k-r})$ and $(\theta_{k+r}, \theta_{k+r+1})$ for $1 \leqslant r \leqslant n-1$. We have therefore found at least $2n(2m+1)$ zeros of $d'_{k,2m}(\theta)$ in $(\theta_{k-n}, \theta_{k+n}]$, and so, because $d'_{k,2m}(\theta)$ has order no greater than $n(2m+1)$, we have located all the zeros of $d'_{k,2m}(\theta)$. Hence $d_{k,2m}(\theta)$ is of constant sign on $(\theta_{k-1}, \theta_{k+1})$, and on each of the intervals $(\theta_{k-r-1}, \theta_{k-r})$ and $(\theta_{k+r}, \theta_{k+r+1})$ for $1 \leqslant r \leqslant n-1$. Now, $d_{k,2m}(\theta_k)=+1$, and since $d_{k,2m}(\theta)$ has zeros of odd multiplicity at $\theta_{k\pm j}$, $1 \leqslant j \leqslant n-1$, it follows that for $0 \leqslant r \leqslant n-1$, we have

$$\operatorname{sgn}(d_{k,2m}(\theta)) = (-1)^r, \qquad \theta_{k-r-1} < \theta < \theta_{k-r}, \quad \theta_{k+r} < \theta < \theta_{k+r+1}.$$

Thus, in the intervals $(-\pi/(2n), \pi/(2n)) = (\theta_0, \theta_1)$ and $(2\pi - \pi/(2n), 2\pi + \pi/(2n)) = (\theta_{2n}, \theta_{2n+1})$,

$$\operatorname{sgn}(d_{k,2m}(\theta)) = \begin{cases} (-1)^{k-1}, & 1 \leqslant k \leqslant n \\ (-1)^k, & n+1 \leqslant k \leqslant 2n \end{cases} \tag{2.7}$$

and so, by (2.7) and Lemma 1,

$$\Delta_{2m,n} = \max\{t_{2m,n}(\theta) \colon |\theta| \leqslant \pi/(2n)\}.$$

(ii) Put

$$g_{k,2m}(\theta) = d_{2n-k+1,2m}(-\theta), \qquad k = 1, 2, ..., 2n.$$

Then each $g_{k,2m}(\theta)$ is in $\mathscr{T}_{n(2m+1),2}$, and

$$\begin{aligned} g^{(t)}_{k,2m}(\theta_j) &= (-1)^t\, d^{(t)}_{2n-k+1,2m}(-\theta_j) \\ &= (-1)^t\, d^{(t)}_{2n-k+1,2m}(\theta_{2n-j+1}) \\ &= \delta_{0,t}\delta_{k,j}, \qquad 1 \leqslant j \leqslant 2n, \quad 0 \leqslant t \leqslant 2m. \end{aligned}$$

Thus $g_{k,2m}(\theta) \equiv d_{k,2m}(\theta)$, or

$$d_{k,2m}(-\theta) \equiv d_{2n-k+1,2m}(\theta), \tag{2.8}$$

and so, by (2.5), it follows that $t_{2m,n}(\theta)$ is an even function. This completes the proof of Lemma 2. ∎

LEMMA 3. $\Delta_{2m,n} = t_{2m,n}(0)$.

Proof. Let $\bar{\theta} \in [-\pi/(2n), \pi/(2n)]$ be such that $\Delta_{2m,n} = t_{2m,n}(\bar{\theta})$. We need to show that $\bar{\theta} = 0$.

First, suppose $\bar{\theta} = \pm\pi/(2n)$. Then, by (2.2) and (2.5), $t_{2m,n}(\bar{\theta}) = t_{2m,n}(\theta_1) = 1$, and so $\Delta_{2m,n} = 1$. Consider

$$r_{2m,n}(\theta) = \sum_{k=1}^{2n} d_{k,2m}(\theta),$$

which satisfies

$$r_{2m,n}^{(t)}(\theta_j) = \delta_{0,t}, \qquad 1 \leqslant j \leqslant 2n, \quad 0 \leqslant t \leqslant 2m.$$

Thus $u_{2m,n}(\theta) := 1 - r_{2m,n}(\theta)$ is a trigonometric polynomial of order $n(2m+1)$ or less which has zeros of multiplicity $2m+1$ at $\theta_1, \theta_2, ..., \theta_{2n}$, and no other zeros in $[0, 2\pi]$, or else is identically zero. In the former case, $u_{2m,n}(\theta)$ changes sign at each of the θ_j, and so, in particular, there exists θ^* close to θ_1 so that $u_{2m,n}(\theta^*) < 0$ (and hence $r_{2m,n}(\theta^*) > 1$). But then

$$1 = \Delta_{2m,n} \geqslant D_{2m,n}(\theta^*) = \sum_{k=1}^{2n} |d_{k,2m}(\theta^*)| \geqslant \sum_{k=1}^{2n} d_{k,2m}(\theta^*) = r_{2m,n}(\theta^*) > 1,$$

which gives a contradiction. On the other hand, if $u_{2m,n}(\theta) \equiv 0$, then $r_{2m,n}(\theta) \equiv 1$, and so for all θ we have

$$1 \geqslant D_{2m,n}(\theta) = \sum_{k=1}^{2n} |d_{k,2m}(\theta)| \geqslant \sum_{k=1}^{2n} d_{k,2m}(\theta) = r_{2m,n}(\theta) = 1,$$

which implies that $D_{2m,n}(\theta) \equiv 1$. However, $D_{2m,n}(\theta)$ and $t_{2m,n}(\theta)$ coincide on (θ_0, θ_1), so $t_{2m,n}(\theta) \equiv 1$. For $n \geqslant 2$ this provides a contradiction, since $t_{2m,n}(\theta_2) = -1$. We conclude that $\bar{\theta} \neq \pm\pi/(2n)$ if $n \geqslant 2$, while if $n = 1$ and $\bar{\theta} = \pm\pi/(2n)$, then $t_{2m,n}(\theta) \equiv 1$.

Second, suppose $0 < |\bar{\theta}| < \pi/(2n)$. Because $t_{2m,n}(\theta)$ is even, we have $t_{2m,n}(\bar{\theta}) = t_{2m,n}(-\bar{\theta})$, and so $t'_{2m,n}(\theta)$ has a zero between $-\bar{\theta}$ and $\bar{\theta}$, in addition to $t'_{2m,n}(\bar{\theta}) = t'_{2m,n}(-\bar{\theta}) = 0$, for a total of at least 3 zeros of $t'_{2m,n}(\theta)$ in (θ_0, θ_1),. Further, by (2.4) we have $d_{k,2m}(\theta + \pi) \equiv d_{k-n,2m}(\theta)$, $n+1 \leqslant k \leqslant 2n$, and $d_{k,2m}(\theta + \pi) \equiv d_{k+n,2m}(\theta)$, $1 \leqslant k \leqslant n$, so that (by (2.5)),

$$t_{2m,n}(\theta + \pi) = (-1)^{n+1}\, t_{2m,n}(\theta).$$

Thus $t'_{2m,n}(\theta)$ has at least 3 zeros in (θ_n, θ_{n+1}). Now, by (2.2) and (2.5), $t'_{2m,n}(\theta)$ has zeros of order at least $2m$ at $\theta_1, \theta_2, ..., \theta_{2n}$. As well, from

$$t_{2m,n}(\theta_j) = \begin{cases} (-1)^{j-1}, & 1 \leqslant j \leqslant n, \\ (-1)^j, & n+1 \leqslant j \leqslant 2n, \end{cases}$$

we see that $t_{2m,n}(\theta)$ has an absolute minimum point θ_1^* in (θ_1, θ_3), so $t'_{2m,n}(\theta)$ has a zero of odd multiplicity at θ_1^*. If $\theta_1^* = \theta_2$, then the multiplicity of the zero of $t'_{2m,n}(\theta)$ at θ_2 must be at least $2m+1$, while if $\theta_1^* \neq \theta_2$, $t'_{2m,n}(\theta)$ has a zero of multiplicity at least 1 at θ_1^*. In either case, we have identified an additional zero of $t'_{2m,n}(\theta)$. Similarly, there is an additional zero of $t'_{2m,n}(\theta)$, corresponding to a maximum point of $t_{2m,n}(\theta)$, in (θ_2, θ_4). Continuing in this fashion, we are able to identify a total of $2n-4$ additional zeros of $t'_{2m,n}\theta)$ in $(\theta_0, \theta_{2n}]$, one in each of the intervals (θ_1, θ_3), (θ_2, θ_4), ..., (θ_{n-2}, θ_n), $(\theta_{n+1}, \theta_{n+3})$, ..., $(\theta_{2n-2}, \theta_{2n})$. Thus the trigonometric polynomial $t'_{2m,n}(\theta)$, which is of order no greater than $n(2m+1)$, has at least $2n(2m+1)+2$ zeros in $(\theta_0, \theta_{2n}]$, and so is identically zero. Hence $t_{2m,n}(\theta)$ is constant. However, for $n \geqslant 2$, $t_{2m,n}(\theta_1)=1$ and $t_{2m,n}(\theta_2)=-1$, so we have a contradiction. Thus $\bar{\theta}$ does not satisfy $0<|\bar{\theta}|<\pi/(2n)$ if $n \geqslant 2$, while if $n=1$ and $0<|\bar{\theta}|<\pi/(2n)$, then $t_{2m,n}(\theta) \equiv 1$.

The above results have established that $\bar{\theta}$ has the unique value of 0 if $n \geqslant 2$, while for $n=1$, either $t_{2m,n}(\theta) \equiv 1$ or $\bar{\theta}$ has the unique value of 0. In all cases, then, we have $\Delta_{2m,n} = t_{2m,n}(0) = D_{2m,n}(0)$. This completes the proof of Lemma 3. ▌

To complete the proof of Theorem 1, we show that

$$\Lambda_{2m,n} = \lambda_{2m,n}(1) = \Delta_{2m,n}.$$

By (2.8), $d_{k,2m}(\theta) + d_{2n-k+1,2m}(\theta)$ is an even function, and so is a cosine polynomial of degree $n(2m+1)-1$ or less. If $x=\cos\theta$, it follows that for $k=1, 2, ..., n$,

$$q_{k,2m}(x) = d_{k,2m}(\theta) + d_{2n-k+1,2m}(\theta) \tag{2.9}$$

is an algebraic polynomial in x of degree $n(2m+1)-1$ or less. Further, for $j=1, 2, ..., n$, and x_j given by (1.17), we have (by (2.2)),

$$q_{k,2m}(x_j) = d_{k,2m}(\theta_j) + d_{2n-k+1,2m}(\theta_j) = \delta_{k,j}.$$

Differentiating (2.9) with respect to θ gives

$$-\sin\theta \, q'_{k,2m}(x) = d'_{k,2m}(\theta) + d'_{2n-k+1,2m}(\theta),$$

and so (again by (2.2)), $q'_{k,2m}(x_j)=0$. Continuing in this fashion, we obtain

$$q^{(t)}_{k,2m}(x_j) = \delta_{0,t}\delta_{k,j}, \qquad 1 \leqslant k, j \leqslant n, \quad 0 \leqslant t \leqslant 2m.$$

Thus, by the uniqueness property of the fundamental polynomials $A_{k,2m,n}(x)$ for $(0, 1, 2, ..., 2m)$ HF interpolation on the Chebyshev nodes, we conclude that

$$q_{k,2m}(x) \equiv A_{k,2m,n}(x), \qquad 1 \leqslant k \leqslant n.$$

Thus, for $-1 \leqslant x \leqslant 1$, we have

$$\lambda_{2m,n}(x) = \sum_{k=1}^{n} |A_{k,2m,n}(x)| = \sum_{k=1}^{n} |d_{k,2m}(\theta) + d_{2n-k+1,2m}(\theta)|$$
$$\leqslant \sum_{k=1}^{2n} |d_{k,2m}(\theta)| = D_{2m,n}(\theta) \leqslant \Delta_{2m,n}. \tag{2.10}$$

On the other hand, consider

$$\lambda_{2m,n}(1) = \sum_{k=1}^{n} |d_{k,2m}(0) + d_{2n-k+1,2m}(0)|.$$

By (2.7), $\operatorname{sgn}(d_{k,2m}(0)) = \operatorname{sgn}(d_{2n-k+1,2m}(0)) = (-1)^{k-1}$, and so

$$\lambda_{2m,n}(1) = \sum_{k=1}^{n} (-1)^{k-1} (d_{k,2m}(0) + d_{2n-k+1,2m}(0))$$
$$= \sum_{k=1}^{n} (-1)^{k-1} d_{k,2m}(0) - \sum_{k=n+1}^{2n} (-1)^{k-1} d_{k,2m}(0)$$
$$= t_{2m,n}(0) = \Delta_{2m,n}. \tag{2.11}$$

From (2.10) we conclude that

$$\Lambda_{2m,n} = \max_{-1 \leqslant x \leqslant 1} \lambda_{2m,n}(x) = \lambda_{2m,n}(1),$$

and so Theorem 1 is established. Also, for future reference, we observe that (2.11) can be written as

$$\Lambda_{2m,n} = \sum_{k=1}^{n} (-1)^{k-1} A_{k,2m,n}(1). \tag{2.12}$$

3. Proof of Theorem 2

To begin, consider the arbitrary system of nodes $X = \{x_{k,n}\}$, as given by (1.1) and (1.2), and define

$$\omega_n(X, x) = \prod_{k=1}^{n} (x - x_{k,n}).$$

The polynomials $A_{k,0,n}(X, x)$, which are the fundamental polynomials for Lagrange interpolation on X, can be written as

$$A_{k,0,n}(X, x) := l_{k,n}(X, x) = \frac{\omega_n(X, x)}{\omega_n'(X, x_{k,n})(x - x_{k,n})}, \qquad 1 \leq k \leq n.$$

Then an explicit formula for the fundamental polynomials $A_{k,m,n}(X, x)$ for $(0, 1, 2, 3, ..., m)$ HF interpolation on X is

$$A_{k,m,n}(X, x) = (l_{k,n}(X, x))^{m+1} \sum_{i=0}^{m} h_{i,k,m,n}(x - x_{k,n})^i, \qquad 1 \leq k \leq n, \quad (3.1)$$

where the coefficients $h_{i,k,m,n}$ (which also depend on X) can be determined from (1.3) (c.f. Vértesi [21] or Sakai and Vértesi [15]).

Now let $m \geq 0$ be fixed, and assume the nodes of interpolation are the Chebyshev nodes $x_k = x_{k,n} = \cos\theta_k$, $k = 1, 2, ..., n$, where $\theta_k = \theta_{k,n} = (2k-1)\pi/(2n)$. By Sakai and Vértesi, [15, Theorem 3.3] and [16, Theorem 5.5], we have

$$h_{i,k,2m,n} = O(1)\left(\frac{n}{\sin\theta_k}\right)^i, \qquad 0 \leq i \leq 2m, \quad 1 \leq k \leq n, \quad n = 1, 2, ..., \quad (3.2)$$

where, both here and subsequently, the $O(1)$ term is uniform in i, k and n. (That is, if the $O(1)$ term is denoted by $C_{i,k,2m,n}$, then there exists a positive number C so that $|C_{i,k,2m,n}| \leq C$ for all values of i, k and n.) Also, if $T_n(x)$ denotes the Chebyshev polynomial $T_n(x) := \cos n(\arccos x)$, $-1 \leq x \leq 1$, whose zeros are the Chebyshev nodes $\{x_k\}$, then

$$l_{k,n}(T, x) = \frac{\omega_n(T, x)}{\omega_n'(T, x_k)(x - x_k)} = \frac{T_n(x)}{T_n'(x_k)(x - x_k)} = \frac{(-1)^{k-1}\sin\theta_k}{n(x - x_k)} T_n(x).$$

Hence, by (2.12), (3.1) and (3.2), we have

$$\begin{aligned} A_{2m,n} &= \sum_{k=1}^{n} \left(\frac{\sin\theta_k}{n(1-\cos\theta_k)}\right)^{2m+1} \sum_{i=0}^{2m} h_{i,k,2m,n}(1-\cos\theta_k)^i \\ &= \sum_{k=1}^{n} \frac{h_{2m,k,2m,n}}{1-\cos\theta_k}\left(\frac{\sin\theta_k}{n}\right)^{2m+1} \\ &\quad + O(1) \sum_{k=1}^{n} \sum_{i=0}^{2m-1} \left(\frac{\sin\theta_k}{n(1-\cos\theta_k)}\right)^{2m+1-i}. \end{aligned} \quad (3.3)$$

Now

$$\sum_{k=1}^{n} \sum_{i=0}^{2m-1} \left(\frac{\sin \theta_k}{n(1-\cos \theta_k)}\right)^{2m+1-i} = \sum_{i=2}^{2m+1} \sum_{k=1}^{n} \left(\frac{\cot(\theta_k/2)}{n}\right)^i,$$

and because $\theta \cot \theta$ is bounded on $[0, \pi/2]$, we can write

$$\sum_{i=2}^{2m+1} \sum_{k=1}^{n} \left(\frac{\cot(\theta_k/2)}{n}\right)^i = O(1) \sum_{i=2}^{2m+1} \sum_{k=1}^{n} \frac{1}{(n\theta_k)^i}$$

$$= O(1) \sum_{i=2}^{2m+1} \sum_{k=1}^{n} \frac{1}{(2k-1)^i} = O(1).$$

Thus (3.3) gives

$$\Lambda_{2m,n} = \sum_{k=1}^{n} \frac{h_{2m,k,2m,n}}{1-\cos \theta_k} \left(\frac{\sin \theta_k}{n}\right)^{2m+1} + O(1). \tag{3.4}$$

The following estimate for the $h_{2m,k,2m,n}$ is a consequence of a more general result due to Sakai and Vértesi [15, Theorem 3.3].

LEMMA 4. *For $j = 0, 1, 2, \ldots,$ define polynomials $p_j(s)$ of degree j by*

$$p_0(s) = 1, \tag{3.5}$$

$$p_j(0) = 0, \qquad j = 1, 2, 3, \ldots, \tag{3.6}$$

and

$$p_j(s+1) = \frac{1}{2j+1} \sum_{i=0}^{j} \binom{2j+1}{2i} p_i(s). \tag{3.7}$$

(Thus

$$p_j(s) = \frac{1}{2j+1} \sum_{i=0}^{j-1} \binom{2j+1}{2i} \sum_{t=0}^{s-1} p_i(t), \qquad j = 1, 2, \ldots, \quad s = 1, 2, \ldots,$$

and so $p_0(s) = 1$, $p_1(s) = s/3$, $p_2(s) = (5s^2 - 2s)/15$, $p_3(s) = (35s^3 - 42s^2 + 16s)/63$, etc.) Then we can write

$$h_{2m,k,2m,n} = \frac{(-1)^m}{(2m)!} p_m(-(2m+1)) \left(\frac{n}{\sin \theta_k}\right)^{2m} (1 + \varepsilon_k), \qquad 1 \leqslant k \leqslant n, \tag{3.8}$$

where $\varepsilon_k = \varepsilon_{k,n}$ *satisfies*

$$\varepsilon_k = O(1)\left(\frac{1}{n} + \frac{1}{K^2}\right), \qquad 1 \leqslant k \leqslant n, \quad n \geqslant 2, \tag{3.9}$$

and $K = \min(k, n-k+1)$.

On substituting (3.8) and (3.9) in (3.4) we obtain

$$\Lambda_{2m,n} = \frac{(-1)^m p_m(-(2m+1))}{(2m)!} \frac{1}{n} \times \left(\sum_{k=1}^{n} \left[1 + O(1)\left(\frac{1}{n} + \frac{1}{K^2}\right)\right] \cot(\theta_k/2) \right) + O(1). \tag{3.10}$$

Now, $\frac{1}{n}\sum_{k=1}^{n} \cot(\theta_k/2) = \frac{2}{\pi}\log n + O(1)$. (See, for example, Rivlin [12, Section 1.3].) Also, because $\theta \cot \theta$ is bounded on $[0, \pi/2]$, we can write

$$\frac{1}{n}\sum_{k=1}^{n} \frac{1}{K^2}\cot(\theta_k/2) = O(1)\sum_{k=1}^{n} \frac{1}{K^2(2k-1)} = O(1)\sum_{k=1}^{n}\frac{1}{K^2}$$
$$= O(1)\sum_{k=1}^{[(n+1)/2]} \frac{1}{k^2} = O(1).$$

Thus, by (3.10), we have, as $n \to \infty$,

$$\Lambda_{2m,n} = \frac{2}{\pi}\frac{(-1)^m p_m(-(2m+1))}{(2m)!}\log n + O(1). \tag{3.11}$$

To evaluate $p_m(-(2m+1))$ we need the following lemma. At several stages of the proof of the lemma, properties of the generalized Bernoulli polynomials $B_k^{(a)}(x)$ are used—details of these properties can be found in, for example, Luke [9, Section 2.8].

LEMMA 5. *For* $j = 0, 1, 2, \ldots$, *suppose the polynomials* $p_j(s)$, *of degree* j, *are defined by* (3.5), (3.6) *and* (3.7). *Let* $B_k^{(a)}(x)$ *denote the generalized Bernoulli polynomial, defined by*

$$\frac{t^a e^{xt}}{(e^t - 1)^a} = \sum_{k=0}^{\infty} \frac{t^k}{k!} B_k^{(a)}(x), \qquad |t| < 2\pi.$$

Then, for all s,

$$p_j(s) = 2^{2j} B_{2j}^{(-s)}(-s/2), \qquad j = 0, 1, 2, \ldots \tag{3.12}$$

Proof. Since $p_0(s) = B_0^{(a)}(s) = 1$ for all a, s, then (3.12) is true for $j = 0$. We next use induction on s to prove that for $j \geq 1$, the equation (3.12) holds true for $s = 0, 1, 2, \ldots$ From $p_j(0) = B_k^{(0)}(0) = 0$ for all $j, k \geq 1$, it follows that (3.12) is true for all $j \geq 1$ if $s = 0$. Suppose (3.12) holds for all $j \geq 1$ if $s = r$. Then, by (3.7),

$$p_j(r+1) = \frac{1}{2j+1} \sum_{i=0}^{j} \binom{2j+1}{2i} p_i(r) = \frac{1}{2j+1} \sum_{i=0}^{j} \binom{2j+1}{2i} 2^{2i} B_{2i}^{(-r)}(-r/2).$$

Using $B_{2k+1}^{(2x)}(x) = 0$ for $k = 0, 1, 2, \ldots$, then reversing the order of summation, we obtain

$$p_j(r+1) = \frac{(-2)^{2j+1}}{2j+1} \sum_{i=0}^{2j+1} \binom{2j+1}{i} \left(-\frac{1}{2}\right)^i B_{2j+1-i}^{(-r)}(-r/2).$$

However (Luke [9, p. 21]), for all k, a, x, y,

$$B_k^{(a)}(x+y) = \sum_{i=0}^{k} \binom{k}{i} B_{k-i}^{(a)}(y)\, x^i,$$

and so

$$p_j(r+1) = \frac{(-2)^{2j+1}}{2j+1} B_{2j+1}^{(-r)}(-(r+1)/2). \tag{3.13}$$

Now, by Luke [9, p. 20], for all k, a, x,

$$aB_k^{(a+1)}(x) = (a-k)\, B_k^{(a)}(x) + k(x-a)\, B_{k-1}^{(a)}(x).$$

Applying this formula to the right-hand side of (3.13), and using $B_{2j+1}^{(-(r+1))}(-(r+1)/2) = 0$, gives

$$p_j(r+1) = 2^{2j} B_{2j}^{(-(r+1))}(-(r+1)/2),$$

as required. Thus, for fixed j, (3.12) holds true for $s = 0, 1, 2, \ldots$ Since both sides of (3.12) are polynomials, we conclude that (3.12) is valid for all real s. ∎

Now, by (3.12), we can write $p_m(-(2m+1)) = 2^{2m} B_{2m}^{(2m+1)}((2m+1)/2)$. However (Luke [9, p. 20]), for all k, x, we have $B_k^{(k+1)}(x) = (-1)^k \prod_{r=1}^{k} (r-x)$. Thus

$$p_m(-(2m+1)) = (-1)^m\, 2^{2m} \prod_{r=1}^{m} (r-1/2)^2 = (-1)^m \left(\frac{(2m)!}{2^m m!}\right)^2.$$

Substituting this result into (3.11) gives the desired (1.11).

4. Proof of Theorem 3

The formula (2.12) gives

$$A_{2,n} = \sum_{k=1}^{n} (-1)^{k-1} A_{k,2,n}(1).$$

Now, the $A_{k,2,n}(x)$ are given by

$$A_{k,2,n}(x) = \frac{(-1)^{k-1} (T_n(x))^3 (1-x_k^2)^{1/2}}{2^{n3}(x-x_k)^3} \times \{2(1-xx_k) - x_k(x-x_k) + (n^2-1)(x-x_k)^2\}$$

(see Byrne, Mills and Smith [2, Section 1]). Since $T_n(1)=1$, we obtain (for $\theta_k = \theta_{k,n} = (2k-1)\pi/(2n)$),

$$A_{2,n} = \frac{1}{2n^3} \sum_{k=1}^{n} \left(\frac{\sin\theta_k}{(1-\cos\theta_k)^2} + n^2 \frac{\sin\theta_k}{1-\cos\theta_k} \right)$$
$$= \frac{1}{2n^3} \sum_{k=1}^{n} \left(\frac{\cot(\theta_k/2)\csc^2(\theta_k/2)}{2} + n^2 \cot(\theta_k/2) \right),$$

and hence

$$A_{2,n} = \frac{2n^2+1}{4n^3} \sum_{k=1}^{n} \cot(\theta_k/2) + \frac{1}{4n^3} \sum_{k=1}^{n} \cot^3(\theta_k/2).$$

From $\cot^3 x = \frac{1}{2}\frac{d^2}{dx^2}(\cot x) - \cot x$, and $\frac{1}{n}\sum_{k=1}^{n} \cot(\theta_k/2) = A_{0,n}$ (see, for example, Rivlin [12, Section 1.3]), it follows that

$$A_{2,n} = \frac{1}{2} A_{0,n} + \frac{1}{8n^3} \sum_{k=1}^{n} \frac{d^2}{dx^2}(\cot x)\Big]_{x=\theta_k/2}$$
$$:= \frac{1}{2} A_{0,n} + S. \tag{4.1}$$

By Shivakumar and Wong's result (1.12) the asymptotic behaviour of $A_{0,n}$ is known, so we need to look carefully at the summation term S on the right-hand side of (4.1).

From the well-known expansion

$$\cot x = \frac{1}{x} - \sum_{t=1}^{\infty} \frac{2^{2t} |B_{2t}|}{(2t)!} x^{2t-1}, \qquad |x| < \pi,$$

we obtain

$$S=\frac{16}{\pi^3}\sum_{k=1}^{n}\frac{1}{(2k-1)^3}-\frac{8}{\pi^3}\sum_{t=2}^{\infty}\frac{|B_{2t}|}{2t(2t-3)!}\left(\frac{\pi}{2n}\right)^{2t}\sum_{k=1}^{n}(2k-1)^{2t-3}$$
$$:=S_1-S_2. \tag{4.2}$$

Shivakumar and Wong [18, Examples 1 and 3] have established the following asymptotic results.

LEMMA 6. *If α is any real number, not -1 or a positive integer, then, as $n\to\infty$,*

$$\sum_{k=1}^{n}(2k-1)^{\alpha}\sim(1-2^{\alpha})\,\zeta(-\alpha)+\frac{(2n)^{\alpha+1}}{\alpha+1}\sum_{s=0}^{\infty}\binom{\alpha+1}{2s}(1-2^{2s-1})\frac{B_{2s}}{(2n)^{2s}}, \tag{4.3}$$

where, if r is a positive integer such that $r>(\alpha+1)/2$, the error due to truncation of the infinite series at the term $s=r-1$ is bounded in absolute value by, and has the same sign as, the first neglected term.

If β is a positive integer, then for any $r=1, 2, ..., [\beta/2]+1$, we have

$$\sum_{k=1}^{n}(2k-1)^{\beta}=\frac{(2n)^{\beta+1}}{\beta+1}\sum_{s=0}^{r-1}\binom{\beta+1}{2s}(1-2^{2s-1})\frac{B_{2s}}{(2n)^{2s}}+\varepsilon_r^{(\beta)}(n), \tag{4.4}$$

where $\varepsilon_r^{(\beta)}(n)=0$ if $r=[\beta/2]+1$, and

$$0\leqslant(-1)^r\,\varepsilon_r^{(\beta)}(n)\leqslant\frac{(2n)^{\beta+1}}{\beta+1}\binom{\beta+1}{2r}(2^{2r-1}-1)\frac{|B_{2r}|}{(2n)^{2r}}$$

if $r<[\beta/2]+1$.

By (4.3), and the remarks concerning the error in trunction, we can write

$$S_1=\frac{14}{\pi^3}\zeta(3)+\frac{2}{n^2\pi^3}\sum_{s=0}^{r-1}(2s+1)(2^{2s-1}-1)\frac{B_{2s}}{(2n)^{2s}}+U_r(n), \tag{4.5}$$

where

$$0\leqslant(-1)^{r+1}\,U_r(n)\leqslant\frac{2}{n^2\pi^3}(2r+1)(2^{2r-1}-1)\frac{|B_{2r}|}{(2n)^{2r}}, \tag{4.6}$$

so it remains to consider the term S_2 in (4.2).

Fix $R \geqslant 1$, and for $t=2, 3, \ldots$, put $\beta = 2t-3$ and

$$r = r(t) = \begin{cases} t-1, & t=2, 3, \ldots, R, \\ R, & t=R+1, R+2, \ldots, \end{cases}$$

in (4.4). This gives

$$\sum_{k=1}^{n} (2k-1)^{2t-3} = \frac{(2n)^{2t-2}}{2t-2} \sum_{s=0}^{r(t)-1} \binom{2t-2}{2s} (1-2^{2s-1}) \frac{B_{2s}}{(2n)^{2s}} + \varepsilon_{r(t)}^{(2t-3)}(n), \tag{4.7}$$

where

$$\begin{cases} \varepsilon_{r(t)}^{(2t-3)}(n) = 0, \quad 2 \leqslant t \leqslant R+1, \\ 0 \leqslant (-1)^R \varepsilon_{r(t)}^{(2t-3)}(n) \leqslant \dfrac{(2n)^{2t-2}}{2t-2} \dbinom{2t-2}{2R} (2^{2R-1}-1) \dfrac{|B_{2R}|}{(2n)^{2R}}, \quad t \geqslant R+2. \end{cases} \tag{4.8}$$

Now substitute (4.7) into the formula for S_2, so

$$\begin{aligned} S_2 = {} & \frac{8}{\pi^3} \sum_{t=2}^{R+1} \frac{|B_{2t}|}{2t(2t-2)!} \frac{\pi^{2t}}{4n^2} \sum_{s=0}^{t-2} \binom{2t-2}{2s} (1-2^{2s-1}) \frac{B_{2s}}{(2n)^{2s}} \\ & + \frac{8}{\pi^3} \sum_{t=R+2}^{\infty} \frac{|B_{2t}|}{2t(2t-2)!} \frac{\pi^{2t}}{4n^2} \sum_{s=0}^{R-1} \binom{2t-2}{2s} \\ & \times (1-2^{2s-1}) \frac{B_{2s}}{(2n)^{2s}} + V_R(n), \end{aligned} \tag{4.9}$$

where

$$V_R(n) = \frac{8}{\pi^3} \sum_{t=R+2}^{\infty} \frac{|B_{2t}|}{2t(2t-3)!} \left(\frac{\pi}{2n}\right)^{2t} \varepsilon_{r(t)}^{(2t-3)}(n). \tag{4.10}$$

On replacing R with r and interchanging the orders of summation in (4.9), we have

$$S_2 = \frac{8}{\pi^3} \sum_{s=0}^{r-1} \frac{(1-2^{2s-1}) B_{2s}}{4n^2(2n)^{2s}} \sum_{t=s+2}^{\infty} \frac{\pi^{2t} |B_{2t}|}{2t(2t-2)!} \binom{2t-2}{2s} + V_r(n).$$

Now, by Shivakumar and Wong [18, equation (3.12)],

$$\sum_{t=s+2}^{\infty} \frac{\pi^{2t} |B_{2t}|}{2t(2t-2)!} \binom{2t-2}{2s} = \frac{\pi^{2s+2}}{(2s)!\,(s+1)} (2^{2s+1}-1) |B_{2s+2}| - (2s+1).$$

Therefore,

$$S_2 = -\frac{2}{n^2\pi^3} \sum_{s=0}^{r-1} \frac{(2^{2s-1}-1)\,B_{2s}}{(2n)^{2s}} \times \left(\frac{\pi^{2s+2}(2^{2s+1}-1)\,|B_{2s+2}|}{(2s)!\,(s+1)} - (2s+1) \right) + V_r(n), \tag{4.11}$$

where (by (4.8) and (4.10)),

$$0 \leqslant (-1)^r\, V_r(n) \leqslant \frac{2}{n^2\pi^3} \frac{(2^{2r-1}-1)\,|B_{2r}|}{(2n)^{2r}} \left(\frac{\pi^{2r+2}(2^{2r+1}-1)\,|B_{2r+2}|}{(2r)!\,(r+1)} - (2r+1) \right). \tag{4.12}$$

Next, on substituting (4.5), (4.6), (4.11) and (4.12) into (4.2), we obtain

$$S = \frac{14}{\pi^3}\zeta(3) - \frac{1}{6\pi n^2} + \frac{8}{\pi^3} \sum_{s=2}^{r} \frac{(-1)^s\,(2^{2s-3}-1)(2^{2s-1}-1)\,|B_{2s-2}|\,|B_{2s}|}{s(2s-2)!} \frac{\pi^{2s}}{(2n)^{2s}} + W_r(n), \tag{4.13}$$

where

$$0 \leqslant (-1)^{r+1}\, W_r(n) \leqslant \frac{8}{\pi^3} \frac{(2^{2r-1}-1)(2^{2r+1}-1)\,|B_{2r}|\,|B_{2r+2}|}{(r+1)(2r)!} \frac{\pi^{2r+2}}{(2n)^{2r+2}}. \tag{4.14}$$

The statement (1.14) of Theorem 3 now follows if we substitute (1.12) and (4.13) into (4.1), and define $\Psi_r(n) = \frac{1}{2}\Phi_r(n) + W_r(n)$. To complete the proof of the theorem, we need to establish the error estimate (1.16) for $\Psi_r(n)$.

To begin, for $s = 2, 3, \ldots$, consider the term

$$D_s := \frac{2s(2s-1)}{\pi^2}(2^{2s-3}-1)\,|B_{2s-2}| - \frac{(2^{2s-1}-1)}{4}\,|B_{2s}|,$$

which appears in the expression (1.15) for C_s. Since $|B_{2s}| = 2((2s)!/(2\pi)^{2s}) \sum_{t=1}^{\infty} t^{-2s}$ (see Luke [9, p. 23]), we have

$$D_s = \frac{(2s)!}{(2\pi)^{2s}} \left[8(2^{2s-3}-1) \sum_{t=1}^{\infty} t^{-2s+2} - \frac{1}{2}(2^{2s-1}-1) \sum_{t=1}^{\infty} t^{-2s} \right]$$
$$> \frac{(2s)!}{(2\pi)^{2s}} \left[8 \times 2^{2s-4} \sum_{t=1}^{\infty} t^{-2s+2} - 2^{2s-2} \sum_{t=1}^{\infty} t^{-2s+2} \right]$$
$$= \frac{2^{2s-2}(2s)!}{(2\pi)^{2s}} \sum_{t=1}^{\infty} t^{-2s+2} > 0$$

(so $C_s > 0$). Now, from (1.13) and (4.14) it follows that

$$-\frac{(2^{2r+1}-1)}{4} |B_{2r+2}| \leqslant (-1)^{r+1} \frac{\pi}{8} \left(\frac{2n}{\pi}\right)^{2r+2} \frac{(r+1)(2r+2)!}{(2^{2r+1}-1)\,|B_{2r+2}|} \Psi_r(n)$$
$$\leqslant \frac{(2r+2)(2r+1)}{\pi^2} (2^{2r-1}-1)\,|B_{2r}|. \tag{4.15}$$

By the positivity of D_{r+1} we have

$$-\frac{(2r+2)(2r+1)}{\pi^2} (2^{2r-1}-1)\,|B_{2r}| < -\frac{(2^{2r+1}-1)}{4} |B_{2r+2}|,$$

and substituting this inequality into the left-hand side of (4.15) yields the desired (1.16).

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